MA10209 – Week 4 Tutorial

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Top Tips (response to sheet 3)

- Talk about injective maps, not injective sets.
  - We don't have a definition for 'X injective into Y' or similar statements, so stick with 'There exists an injective map f between sets X and Y.'

- Proofs use definitions and theorems rather than intuition.
  - e.g. Intuitively, countable means you can (in some sense) count the elements. Sheet 3 Q1 is about showing the mathematical definition does what you expect it to, so using intuition here isn't good enough. You need to prove it from the definition.

Top Tips (response to sheet 3)

- Don't skip parts of the question.
  - 'Discuss whether the following relations are reflexive, symmetric or transitive' is an instruction to give a yes/no answer and a proof/counterexample for each of the three.

- Don't give things the same symbol unless you know they're the same.
  - For example,

$$\text{If } x \sim y \iff x = ny,$$
$$\text{and we know that } a \sim b, b \sim c,$$
$$\text{then } a = n_1 b, b = n_2 c.$$

Top Tips (response to sheet 3)

- If you use a concept you haven't defined in lectures, define it in each question.
- If you use a statement you haven't proved in lectures or on a previous sheet, you'll need to prove it when you use it.
  - Be especially careful if using the internet for ideas – different universities present algebra topics in different orders. Because of this, some of the proofs you'll find use concepts you might not have met yet.
  - Make sure you fully understand every step of your answer! If asked to reproduce the argument in the tutorial, could you?

Key concepts

- Primes
- Coprime
- Pigeonhole principle
- Greatest common divisor
- Lowest common multiple

Primes – true/false?

- All prime numbers are of the form $4n-1$ or $4n+1$.
- All primes $p \geq 3$ are of the form $4n-1$ or $4n+1$.
- The sum of two primes cannot be prime.

Pigeonhole principle

- If we have n objects in m boxes, with $n>m$,
  then there must be at least one box containing more than one object.
- Common sense!

- Some examples:
  - There must be two people in studying maths at Bath who share the same birthday.
  - In a tournament where each team meets every other team once, at all points in the tournament, there are two teams that played the same number of games.

GCD/LCM

- Let p_i be the i[th] prime.

- Then using the fundamental theorem of arithmetic, we can write any natural numbers m, n as
  $m = \prod_{i=1}^{k} p_i^{a_i}$ for $a_i \in \mathbb{N}$ for all $i \in \{1, 2, \ldots, k\}$.
  $n = \prod_{i=1}^{k} p_i^{b_i}$ for $b_i \in \mathbb{N}$ for all $i \in \{1, 2, \ldots, k\}$.
  for some natural number k.

- How can the GCD and LCM be written in this case?

GCD / LCM

$$\gcd(m,n) = \prod_{i=1}^{k} p_i^{c_i},$$
$$\operatorname{lcm}(m,n) = \prod_{i=1}^{k} p_i^{d_i},$$

$$\text{where } c_i = ? \text{ and } d_i = ?$$

Exercise Sheet 4 - overview

- Nastiest sheet yet! ☺
- Don’t get caught up in trying to do the questions in any specific order. Read all the questions and if you have any ideas, get them down on paper.
- DON’T go to the internet for these answers (except where directed) – you won’t learn anything by copying this sheet’s answers off a website.
- DO talk to friends about ideas and concepts.

Exercise Sheet 4 - overview

- Q1 – what happens if a prime number is not of the specified form?
  - *When you have eliminated the impossible, whatever remains, however improbable, must be the truth.* Sherlock Holmes
- Q2 – tricky, but not impossible. Use the hint! ☺
- Q3 – experiment with small values of n. If you find a pattern, try to prove that
  - (a) if you can put the water into one glass, then *n* must be of the specified form, and
    (b) if *n* is of the specified form, you can put the water into one glass.

Exercise Sheet 4 - overview

- Q4-6: pigeonhole principle
- Q4 – every natural number can be written in the form
  $n = 2^a b$ where *b* is an odd number
  - What must the highest odd factors be for n+1, n+2, …, 2n?
- Q5 – use the hint! ☺
- Q6 – how would you go about finding the *n* smallest non-primes which are coprime?
  - 1 divides everything, so for *k* a natural number, 1 & *k* are not coprime.

Exercise Sheet 4 - overview

- Q7 – use ideas discussed earlier
- Q8 – note that $f(n)$ is of the same form as the polynomial in the hint (with suitable substitution for x).
  - Proof by induction.
- Q9 – if $m|a$ and $m|b$, the $m|a-b$
  - Proof?